DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **34**, Number **4**, April **2014**

doi:10.3934/dcds.2014.34.1663

pp. **1663–1681**

ON THE LOCAL THEORY OF PRESCRIBED JACOBIAN EQUATIONS

Neil S. Trudinger

Centre for Mathematics and its Applications
Australian National University
Canberra, ACT 0200, Australia

Abstract. We develop the fundamentals of a local regularity theory for prescribed Jacobian equations which extend the corresponding results for optimal transportation equations. In this theory the cost function is extended to a *generating function* through dependence on an additional scalar variable. In particular we recover in this generality the local regularity theory for potentials of Ma, Trudinger and Wang, along with the subsequent development of the underlying convexity theory.

1. Introduction.

Let Ω be a domain in Euclidean n-space, $\mathbb{R}^n$, and Y a mapping from $\Omega \times \mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$. The *prescribed Jacobian equation*, PJE, is a partial differential equation of the form,

$$\det DY(\cdot, u, Du) = \psi(\cdot, u, Du), \tag{1.1}$$

where ψ is a given scalar function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and Du denotes the gradient of the function $u : \Omega \to \mathbb{R}$. Denoting points in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ by (x, z, p), we see that the special case,

$$Y(x, z, p) = p, \tag{1.2}$$

corresponds to the standard Monge-Ampère equation,

$$\det D^2u = \psi(\cdot, u, Du). \tag{1.3}$$

We will always assume that the matrix Y_p is invertible, that is $\det Y_p \neq 0$, whence we may write (1.1) as a general equation of Monge-Ampère type,

$$\det[D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du), \tag{1.4}$$

where

$$A = -Y_p^{-1}(Y_x + Y_z \otimes p), \quad B = (\det Y_p)^{-1}\psi. \tag{1.5}$$

A function $u \in C^2(\Omega)$ is degenerate elliptic, (elliptic), for equation (1.4), henceforth called *admissible*, whenever

$$D^2u - A(\cdot, u, Du) \geq 0, \quad (> 0), \tag{1.6}$$

in Ω. If u is an elliptic solution of (1.4), then the function $B(\cdot, u, Du)$ is positive. Accordingly we will assume throughout that B is at least non-negative in $\Omega\times\mathbb{R}\times\mathbb{R}^n$, that is ψ and $\det Y_p$ have the same sign.

2010 *Mathematics Subject Classification.* Primary: 35J66, 35J25; Secondary: 78A05.

Key words and phrases. Prescribed Jacobian equations, generating function, convexity, duality, existence and regularity.

Research supported by Australian Research Council Grant.

The *second boundary value problem* for the prescribed Jacobian equation is to prescribe the image,

$$Tu(\Omega) := Y(\cdot, u, Du)(\Omega) = \Omega^*, \tag{1.7}$$

where Ω^* is another given domain in $\mathbb{R}^n$. When ψ is separable, in the sense that

$$|\psi(x, z, p)| = f(x)/g \circ Y(x, z, p), \tag{1.8}$$

for positive $f, g \in L^1(\Omega)$, $L^1(\Omega^*)$ respectively, then a necessary condition for the existence of an admissible solution, for which the mapping T is a diffeomorphism, to the second boundary value problem (1.1), (1.7) is the *mass balance* condition,

$$\int_\Omega f = \int_{\Omega^*} g. \tag{1.9}$$

For the standard Monge-Ampère equation (1.3), with $Tu = Du$, the classical solvability of the second boundary value problem, under the mass balance condition (1.9), was proved by Delanöe, ($n = 2$), [4], Caffarelli [2] and Urbas[32], under the hypothesis that both domains, Ω and Ω^* are uniformly convex. As already pointed out in [13], (1.7) implies a nonlinear *oblique* boundary condition. A weaker interpretation of the boundary condition (1.7) arises through optimal transportation, in which case Caffarelli [1] proved that the convexity of the target Ω^* suffices for local smoothness of solutions.

Interest in the general case was stimulated in the last decade through its application to regularity in optimal transportation, ([21, 29]). Here we are given a cost function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the vector field Y is generated by the equation,

$$c_x(x, Y(x, p)) = p, \tag{1.10}$$

which we assume to be uniquely solvable for $p \in c_x(\Omega \times \Omega^*)$, with non-vanishing determinant, that is $\det c_{x,y}(x, y) \neq 0$, for all $(x, y) \in \Omega \times \Omega^*$. In the corresponding Monge-Ampère equation (1.4), we have

$$A(x, z, p) = c_{xx}(x, Y(x, p)), \quad B(x, z, p) = \det c_{x,y}(x, Y(x, p)\psi(x, z, p). \tag{1.11}$$

Conditions for local regularity were found in [21] and for global regularity in [30], with the latter being extended to general prescribed Jacobian equations in [23, 24].

In this article we consider a more general situation where the cost function is replaced by a smooth *generating function* $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, with the resultant vector field Y also depending on u. This enables the corresponding local theory to embrace recent work in near field optics, [11, 12]. The assumptions on the generating function G are parallel to those introduced for cost functions in optimal transportation in [21, 30]. Accordingly we let $\mathcal{U}$ be an open set in $\mathbb{R}^n \times \mathbb{R}^n$ and I an open interval in $\mathbb{R}$. For points $(x, y) \in \mathcal{U}$, we denote their corresponding projections by $\mathcal{U}^*_x = \{y \in \mathbb{R}^n \mid (x, y) \in \mathcal{U}\}$, $\mathcal{U}_y = \{x \in \mathbb{R}^n \mid (x, y) \in \mathcal{U}\}$ and write $\mathcal{U}^{(1)} = \bigcup\{\mathcal{U}_y \mid y \in \mathbb{R}^n\}$ and $\mathcal{U}^{(2)} = \bigcup\{\mathcal{U}^*_x \mid x \in \mathbb{R}^n\}$ Denoting points in I by z, we assume that G is smooth in $\mathcal{U} \times I, G_z \neq 0$ and

G1: For each $(x, y) \in \mathcal{U}$, there exists an open interval $I(x, y) \subset I$ such that the mapping $(G_x, G)(x, \cdot, \cdot)$ is one-to -one in $y \in \mathcal{U}^*_x, z \in I(x, y)$, for each $x \in \mathcal{U}^{(1)}$.

G2: For each $(x, y) \in \mathcal{U}, z \in I(x, y)$, $\det E(x, y, z) \neq 0$, where E is the $n \times n$ matrix given by

$$E = [E_{x,y}] = [G_{x,y} - (G_z)^{-1} G_{x,z} \otimes G_y]. \tag{1.12}$$

From G1 and G2, the vector field Y, together with a scalar function Z, are generated by G through the equations,

$$G_x(x, Y, Z) = p, \quad G(x, Y, Z) = u. \tag{1.13}$$

The significance of the additional function Z will become apparent later. Note that the Jacobian determinant of the mapping $(y, z) \to (G_x, G)(x, y, z)$ is $G_z \det E, \neq 0$ by G2, so that Y and Z are accordingly smooth. Also by differentiating (1.13), with respect to p, we obtain $Y_p = E^{-1}$. Next using (1.5) or differentiating (1.13) for $p = Du$, with respect to x, we obtain that the prescribed Jacobian equation (1.1) is a Monge-Ampère equation of the form (1.4) with

$$\begin{aligned} A(x, u, p) &= G_{xx}[x, Y(x, u, p), Z(x, u, p)], \\ B(x, u, p) &= \det E(x, Y, Z)\psi(x, u, p) \end{aligned} \tag{1.14}$$

and is well defined in domains $\Omega \Subset \mathcal{U}^{(1)}$ for $Y \in \mathcal{U}_x^*, Z \in I(x, Y), x \in \Omega$. Note that the latter restrictions may automatically place constraints on u and Du.

In the optimal transportation case, we have

$$G(x, y, z) = c(x, y) - z, \quad G_z = -1, \quad I = I(x, y) = \mathbb{R}, \tag{1.15}$$

and we recover (1.10) and (1.11) above.

Note also that by adjusting the dependence of G on z, we can fix the interval I, as well as the sign of G_z, as we wish. For convenience with other examples, we will assume either $I = (0, \infty)$ or $I = \mathbb{R}$ as above and assume $G_z < 0$. Let us also denote

$$\Gamma = \Gamma(\mathcal{U}) = \big\{(x, y, z) \in \mathcal{U} \times I \mid z \in I(x, y)\big\}$$

We mention also that the idea behind conditions G1 and G2 is to determine the mapping T from a tangential intersection of the graphs of the functions u and $G(\cdot, y, z)$. If the graph of G is also a local support from below, we obtain the ellipticity condition (1.6), which corresponds to a local convexity. The geometric picture is further amplified in [25], where the theory developed here is already outlined. Knowledge of the defining function G can also lead to a more efficient derivation of the Monge-Ampère equation, using (1.14) rather than computing it directly from (1.5). Also note that when the graph of G is a local support from above we obtain the complementary ellipticity condition,

$$D^2 u - A(\cdot, u, Du) \leq 0, \tag{1.16}$$

corresponding to a local concavity. In this case by setting $u^- = -u, A^-(\cdot, u, p) = -A(\cdot, -u, -p)$, we also obtain a degenerate elliptic equation of the form (1.4) for the function u^-. Furthermore if $n = 2$ and $B > 0$ in (1.4), then either of the strict versions of (1.6) or (1.16) must hold.

For the regularity of weak solutions we need a global convexity theory. To develop this, we in turn need a dual condition, which in the optimal transportation case is simply obtained by interchanging x and y. Using the property that the generating function G is strictly monotone with respect to z, we introduce a dual function $G^* = H$ on $\mathcal{U} \times \mathbb{R}$ by

$$G[x, y, H(x, y, u)] = u. \tag{1.17}$$

Clearly H is well defined whenever $(x, y) \in \mathcal{U}$ and $u \in G(x, y, \cdot)(I)$. Furthermore we have the relations

$$H_x = -G_x/G_z, H_y = -G_y/G_z, H_u = 1/G_z. \tag{1.18}$$

This motivates the following dual condition:

G1*: The mapping $Q := -G_y/G_z$ is one-to-one in x, for all $y \in \mathcal{U}^{(2)}, z \in I(x,y)$.
Writing $J(x,y) = G(x,y,\cdot)I(x,y)$, we see that condition G1* corresponds to G1 with x and y interchanged and $I(x,y)$ replaced by $J(x,y)$. Furthermore the Jacobian matrix of the mapping $x \to Q(x,y,z)$ is $-E/G_z$ so its determinant is automatically non-zero when condition G2 holds. Analogously to Γ above, we may also denote the dual set,

$$\Gamma^* = \big\{(x,y,u) \in \mathcal{U} \times \mathbb{R} \mid u \in J(x,y)\big\}.$$

Our next conditions extend the conditions A3 and A3w introduced for regularity in [21, 30] and are expressed in terms of the matrix function A in (1.4), namely:

G3 (**G3w**) $$A^{kl}_{ij}\xi_i\xi_j\eta_k\eta_l := (D_{p_kp_l}A_{ij})\xi_i\xi_j\eta_k\eta_l > (\geq)\ 0,$$

for all $(x,Y) \in \mathcal{U}, Z \in I(x,Y), \xi,\eta \in \mathbb{R}^n$ such that $\xi.\eta = 0$.

As in [23], we may write equivalently that A is *strictly regular*, (*regular*), in the set

$$\Gamma' = \big\{(x,u,p) \mid x \in \mathcal{U}^{(1)},$$
$$u = G(x,y,z), p = G_x(x,y,z), \text{for some } y \in \mathcal{U}^*_x, z \in I(x,y)\}.$$

For our convexity theory we will also need a condition that the matrix function A is monotone with respect to u, that is

G4 (**G4w**) $$D_uA_{ij}\xi_i\xi_j > (\geq)\ 0,$$

for all $(x,Y) \in \mathcal{U}, Z \in I(x,Y), \xi \in \mathbb{R}^n$.

In this paper we will not use the strict monotonicity G4.

We illustrate the above conditions with the example of a parallel beam from [19]. For $\mathcal{U} = \mathbb{R}^n \times \mathbb{R}^n$ and $I = (0,\infty)$, we define

$$G(x,y,z) = \frac{1}{2z} - \frac{z}{2}|x-y|^2. \tag{1.19}$$

Then G satisfies G1,G2,G1*,G3, G4 with

$$G_z = -\frac{1}{2}(z^{-2} + |x-y|^2) < 0, \quad G_x = -z(x-y), \quad G_y = -z(y-x),$$
$$E = z\big\{(1+z^2|x-y|^2)I - 2z^2(x-y)\otimes(x-y)\big\}/(1+z^2|x-y|^2)),$$
$$\det E = z^n(1-z^2|x-y|^2)/(1+z^2|x-y|^2) > 0, z \in I(x,y),$$
$$Y = x + \frac{2uDu}{(1-|Du|^2)}, \quad Z = \frac{(1-|Du|^2)}{2u}, \quad A = -ZI,$$
$$H = \frac{1}{u + (u^2+|x-y|^2)^{1/2}}, \quad I(x,y) = (0, \frac{1}{|x-y|}), J(x,y) = (0,\infty).$$

The corresponding Monge-Ampère equation,

$$\det\big\{D^2u + \frac{(1-|Du|^2)}{2u}I\big\} = \frac{(1-|Du|^2)^{n+1}}{(1+|Du|^2)(2u)^n}\psi, \tag{1.20}$$

is well defined for $u > 0$ and $|Du| < 1$.

In the underlying physical model, a parallel beam of light directed upwards through $\mathbb{R}^n$ is reflected, in accordance with Snell's law, from the graph of u to a target back in $\mathbb{R}^n$. The restrictions $u > 0$ and $|Du| < 1$ are thus obvious and $Tu(x)$ is the point in $\mathbb{R}^n$ reached by a incoming ray in the upwards direction through the point x. The graph of G in $\mathbb{R}^{n+1}$ is a focal paraboloid in that every vertical ray is reflected to the focus $(y,0)$. When $\psi = f/g(T)$, (1.20) is the PDE satisfied by a function u for which the reflection map Tu pushes forward the density f to the density g.

In the next section, we show that a convexity theory, with respect to generating functions, replicating usual convexity, and more generally the optimal transportation case as developed in [10, 14, 29, 31, 35] can be built under conditions G1, G2, G1*,G3w and G4w. In particular we prove that the local convexity of smooth functions implies their global convexity for appropriately convex domains, that normal mappings are determined by sub-differentials and that sections and contact sets are convex in the extended sense. As well we prove that domain convexity with respect to generating functions is a special case of that determined by vector fields in [23, 24]. Our proofs follow the optimal transportation case, as presented for example in [29], although they also depend on more intricate calculations.

In Section 3, we derive the dual Monge-Ampère equation, satisfied by the G-transform v of a G-convex solution u of (1.1), given by

$$v(y) = u^*_G(y) = \sup_\Omega H(\cdot, y, u). \tag{1.21}$$

and prove that conditions G3 and G3w are invariant under duality.

Section 4 is devoted to the existence and regularity of generalized solutions of the second boundary value problem (1.7), (1.8), with initial and target domains satisfying $\Omega \times \Omega^* \subset \mathcal{U}$. For existence we follow the approach in [3, 11], which corresponds to the existence of potential functions in optimal transportation. Here we need an additional condition to control gradients of solutions, which we may express as:

G5: There exists constants $m_0 \geq -\infty, K_0 \geq 0$, such that $(m_0, \infty) \subset J(x, y)$ and

$$|G_x(x, y, z)| \leq K_0$$

for all $x \in \Omega, y \in \Omega^*, G(x, y, z) > m_0$.

The example (1.18) clearly satisfies G5 for $m_0 = 0, K_0 = 1$, while in the optimal transportation case (1.15), $m_0 = -\infty$. Note that by translation, we may always assume either $m_0 = 0$ or $m_0 = -\infty$. Under condition G5, there exists a generalized solution u with $u(x_0) = u_0$, provided $u_0 > m_0 + K_0 d(x_0, \partial\Omega)$. More generally we can also replace G by composites $\mu(G)$ for suitable smooth functions $\mu \in C^1(m_0, \infty), \mu' \neq 0$. This enables us to embrace more examples such as near field reflection from a point source, as treated in [11, 12]; (see Section 4).

Following [21], we then prove local regularity under conditions G1,G2,G1*G3 and G4w for target domains satisfying the appropriate convexity conditions. We remark that the existence of globally smooth elliptic solutions under corresponding conditions, including stronger domain convexity conditions, follows from the theory of the general prescribed Jacobian equation in [24] and its adaptation to near field reflection problems in [19].

2. Convexity theory.

We begin with the appropriate definitions of convexity with respect to generating functions.

Let Ω be a bounded domain in $\mathcal{U}^{(1)}$, denote $\mathcal{U}^*_\Omega = \cap_{x\in\Omega}\mathcal{U}^*_x$ and let G be a generating function on $\mathcal{U} \times I$, satisfying conditions G1 and G2 , with I an open interval in $\mathbb{R}$ as in the previous section. A function $u \in C^0(\Omega)$ is called *G-convex* in Ω, if for each $x_0 \in \Omega$, there exists $y_0 \in \mathcal{U}^*_\Omega$ and $z_0 \in I(x_0, y_0)$ such that

$$\begin{aligned} u(x_0) &= G(x_0, y_0, z_0), \\ u(x) &\geq G(x, y_0, z_0) \end{aligned} \tag{2.1}$$

for all $x \in \Omega$. If u is differentiable at x_0, then $y_0 = Tu(x_0) = Y(x_0, u(x_0), Du(x_0))$, while if u is twice differentiable at x_0, then

$$D^2u(x_0) \geq D_x^2 G(x_0, y_0, z_0),$$

that is u is admissible for equation (1.4) at x_0. If $u \in C^2(\Omega)$, we call u *locally G-convex* in Ω if this inequality holds for all $x_0 \in \Omega$. We will also refer to functions of the form $G(\cdot, y_0, z_0)$ as *G-affine* and as a *G-support* at x_o if (2.1) is satisfied.

As in the optimal transportation case [21], we also have corresponding notions of domain convexity. There are various possibilities depending on what quantities are fixed. For our purposes here we make the following definitions for domains Ω and Ω^* satisfying $\Omega \times \Omega^* \subset \mathcal{U}$.

The domain Ω is *G-convex* with respect to $y_0 \in \mathcal{U}^*_\Omega$, $z_0 \in I(\Omega, y_0) = \cap_\Omega I(\cdot, y_0)$ if the image $Q_0(\Omega) := -G_y/G_z(\cdot, y_0, z_0)(\Omega)$ is convex in $\mathbb{R}^n$;

The domain Ω^* is *G^*-convex* with respect to (x_0, u_0), where $x_0 \in \mathcal{U}^*_\Omega$ and $u_0 \in J(x_0, \Omega^*) = \cap_{\Omega^*} J(x_0, \cdot)$, if the image $P_0(\Omega^*) := G_x[x_0, \cdot, H(x_0, \cdot, u_0)](\Omega^*)$ is convex in $\mathbb{R}^n$.

Note when we use condition G3w below for convexity results and their consequences, we will assume at least that the convex hulls of the images $Q_0(\Omega)$ and $P_0(Tu(\Omega))$ lie in $Q(\Gamma), G_x(\Gamma)$ respectively.

The second definition is clearly a special case of the notion of Y^*-convexity in [23], as it is equivalent to the set $P_0(\Omega^*) = \{p \in \mathbb{R}^n \mid Y(x_0, u_0, p) \in \Omega^*\}$ being convex. Moreover we will define the domain Ω^* to be G^*-convex with respect to a function $u \in C^0(\Omega)$ if Ω^* is G^*-convex with respect to each point on the graph of u, that is the sets $P_x(\Omega^*) = \{p \in \mathbb{R}^n \mid Y(x, u(x), p) \in \Omega^*\}$ are convex for each $x \in \Omega$ The relationship of the first definition and the notion of Y-convexity in [23] is treated in Lemma 2.4 below. Our main Lemmas 2.1, 2.2 and 2.3 extend corresponding results for optimal transportation in [30, 10, 15] and [5].

Lemma 2.1. *Assume G1,G2,G1*,G3w and G4w hold in $\mathcal{U} \times I$ and that $u \in C^2(\Omega)$ is locally G-convex in Ω. Then if Ω is G-convex with respect to each point in $(Y, Z)(\cdot, u, Du)(\Omega)$, u is G-convex in Ω.*

More generally, we have for any $x_0 \in \Omega, y_0, z_0 = Y, Z(x_0, u(x_0), Du(x_0))$, the G-affine function $G(\cdot, y_0, z_0)$ is a G-support, provided Ω is G-convex with respect to (y_0, z_0).

We defer the proof of Lemma 2.1, until the end of the section and proceed now to consider the corresponding extensions of normal mappings and sections.

Let $u \in C^0(\Omega)$ be G-convex in Ω. We define the *G-normal* mapping of u at $x_0 \in \Omega$ to be the set:

$$Tu(x_0) = \big\{y_0 \in \mathcal{U}_\Omega \mid u(x) \geq G\big(x, y_0, H(x_0, y_0, u_0)\big) \text{ for all } x \in \Omega\big\},$$

where $u_0 = u(x_0)$. Clearly Tu agrees with our previous terminology when u is differentiable and moreover in general

$$Tu(x_0) \subset Y(x_0, u(x_0), \partial u(x_0)),$$

where ∂u denotes the subdifferential of u.

Lemma 2.2. *Assume G1,G2,G1*,G3w,G4w hold in $\mathcal{U} \times I$ and suppose $u \in C^0(\Omega)$ is G-convex in Ω. Then for any $x_0 \in \Omega$, we have*

$$Tu(x_0) = Y(x_0, u(x_0), \partial u(x_0)).$$

For a fixed $y_0 \in Tu(x_0)$, corresponding $z_0 = H(x_0, y_0, u_0)$ and $\sigma > 0$, we define the *G-section*, S_σ by

$$S_\sigma = S_\sigma(x_0, y_0) = \big\{x \in \Omega \mid u(x) < G(x, y_0, z_0) + \sigma\big\}$$

and lower contact set S_0 by

$$S_0 = S_0(x_0, y_0) = \big\{x \in \Omega \mid u(x) = G(x, y_0, z_0)\big\}.$$

Lemma 2.3. *Assume G1,G2,G1*,G3w,G4w hold in $\mathcal{U} \times I$ and suppose $u \in C^0(\Omega)$ is G-convex in Ω, with Ω itself being G-convex with respect to $y_0 \in Tu(x_0)$ and $z_0 = H(x_0, y_0, u(x_0))$. Then the sets S_σ and S_0 are also G-convex with respect to (y_0, z_0).*

The proofs of Lemmas 2.1,2.2 and 2.3 reduce to calculations that are more complicated versions of those which underwrite our starting point in [30, 31], relating domain convexity to the Monge-Ampère equation (1.4). We will adopt the following notation:

$$\begin{aligned} G_{i,\cdots,j,\cdots,z,\cdots,z} &= \frac{\partial}{\partial x_i}\cdots\frac{\partial}{\partial y_j}\cdots\frac{\partial}{\partial z}\cdots\frac{\partial}{\partial z}\,G, \\ E_{i,j} &= E_{x_i,y_j} \\ \big[E^{i,j}\big] &= E^{-1} = \big[D_{p_j}Y^i\big], \end{aligned} \tag{2.2}$$

so that in particular we have the following formulae for differentiation with respect to the p variables:

$$\begin{aligned} Z_p &= -Y_p\,\frac{G_y}{G_z} \\ D_p &= Y_p\big(D_y - \frac{G_y}{G_z}\,D_z\big) \\ D_{p_k} &= E^{r,k}\big(D_{y_r} - \frac{G_{,r}}{G_z}\,D_z\big) \\ D_{p_k}G_{ij} &= E^{r,k}\Big(G_{ij,r} - \frac{G_{,r}}{G_z}\,G_{ij,z}\Big) \end{aligned} \tag{2.3}$$

Next from condition G1*, we infer the existence of a mapping X, defined uniquely by

$$\frac{G_y}{G_z}\Big(X(y,z,q),y,z\Big) = -q \tag{2.4}$$

for all $q \in -\frac{G_y}{G_z}(\cdot, y, z)(\Omega)$. It follows that

$$X_q = -G_z Y_p, \quad E = -G_z X_q^{-1} \tag{2.5}$$

while for fixed y, z we have the following formulae for differentiation with respect to the q variables:

$$\begin{aligned} D_{q_i} &= -G_z E^{i,r} D_{x_r} \\ D^2_{q_\xi q_\xi} &= G_z^2 E^{i,r} E^{j,s} \xi_i \xi_j D_{x_r x_s} + G_z E^{j,s} D_{x_s}\big(G_z E^{i,r}\big)\xi_i\xi_j D_{x_r} \\ &= G_z^2 E^{i,r} E^{j,s} \xi_i \xi_j \Big\{ D_{x_r x_s} + \frac{1}{G_z}\big[E^{k,l}\big(G_z G_{rs,k} - G_{rs,z} G_{,k}\big) D_{x_l} \\ &\qquad - \frac{2}{G_z} G_{s,z} D_{x_r}\big]\Big\} \\ &= G_z^2 E^{i,r} E^{j,s} \xi_i \xi_j \Big\{ D_{x_r x_s} - \big(D_{p_l} G_{rs}\big) D_{x_l} - \frac{2}{G_z} G_{s,z} D_{x_r}\Big\} \\ &= G_z^2 E^{i,r} E^{j,s} \xi_i \xi_j \Big\{ D_{x_r x_s} - \big(D_{p_l} G_{rs}\big) D_{x_l}\Big\} + 2E^{j,s}\xi_j G_{s,z} D_{q_\xi}, \end{aligned} \tag{2.6}$$

for any unit vector, $\xi \in \mathbb{R}^n$ and $q_\xi = q.\xi$.

The proofs of Lemmas 2.1, 2.2 and 2.3 follow from formulae (2.6). First we note also that our definition of G-convex domain aligns with that determined by the vector field Y in [23].

Lemma 2.4. *Assume G1,G2 and G1* hold in $\mathcal{U} \times I$ with $\partial\Omega \in C^2$. Then Ω is G-convex with respect to $y_0 \in \mathcal{U}^*_x, z_0 \in I(\Omega, y_0)$ if and only if*

$$[D_i \gamma_j(x) - G_{ij,p_k}(x, y_0, z_0)\, \gamma_k(x)]\, \tau_i \tau_j \geq 0, \tag{2.7}$$

for all $x \in \partial\Omega$, unit outer normal γ and unit tangent vector τ.

We prove Lemma 2.4 by applying formula (2.6) to the distance function $d = \operatorname{dist}(\cdot, \partial\Omega)$ and using the orthogonality of $Dd = -\gamma$ and τ to cancel the last term in (2.6). □

To prove Lemmas 2.1, 2.2 and 2.3, we let u be locally G-convex in Ω and for a fixed point $x_0, y_0 \in \mathcal{U}, z_0 = H(x_0, y_0, u(x_0))$, define the height function,

$$h(x) = u(x) - G(x, y_0, z_0). \tag{2.8}$$

Making the coordinate transformation, $x \to q = -G_y/G_z(x, y_0, z_0)$, and setting $y = Tu(x) = Y(x, u(x), Du(x)), z = Z(x, u(x), Du(x))$ we compute, using formulae (2.6),

$$\begin{aligned} D_{q_\xi} h &= -G_z E^{i,k} \xi_i D_{x_k} h, \\ D^2_{q_\xi q_\xi} h &= G_z^2 E^{i,k} E^{j,l} [D_{x_k x_l} - D_{p_r} G_{kl} D_{x_r}] h \xi_i \xi_j + 2E^{j,s} G_{s,z} D_{q_i} h \xi_i \xi_j \\ &\geq G_z^2 \big\{ G_{kl}(x, y, z) - G_{kl}(x, y_0, z_0) - D_{p_r} G_{kl}(x, y_0, z_0) \\ &\qquad [G_r(x, y, z) - G_r(x, y_0, z_0)]\big\} E^{i,k} E^{j,l} \xi_i \xi_j + 2E^{j,s} G_{s,z} \xi_j D_{q_\xi} h, \end{aligned} \tag{2.9}$$

for any unit vector, $\xi \in \mathbb{R}^n$. Setting $G_0 = G(x, y_0, z_0), p = G_x(x, y, z), p_0 = G_x(x, y_0, z_0)$ and using condition G4w, we then have for $h \geq 0$, that is for $u \geq G_0$

$$\begin{aligned} D^2_{q_\xi q_\xi} h &\geq G_z^2 [A_{kl}(x, G_0, p) - A_{kl}(x, G_0, p_0) - D_{p_r} A_{kl}(x, G_0, p_0) \\ &\qquad (p_r - p_{0r})] E^{i,k} E^{j,l} \xi_i \xi_j + E^{j,s} G_{s,z} \xi_j D_{q_\xi} h, \\ &\geq \frac{1}{2} D_{p_r p_s} A_{kl}(x, u_0, p^*) D_r h D_s h(x) E^{i,k} E^{j,l} \xi_i \xi_j + E^{j,s} G_{s,z} \xi_j D_{q_\xi} h, \\ &\geq -K |D_{q_\xi} h|, \end{aligned} \tag{2.10}$$

by condition G3w and Taylor's formula, for some p^* on the straight line segment ℓ joining p and p_0 and constant K depending on G,Ω and ℓ. Setting $q_0 = q(x_0, y_0, z_0)$,

$q_t = tq + (1-t)q_0$, $x_t = X(q_t, y_0, z_0$, $0 \le t \le 1$ and defining $h_0(t) = h(x_t)$), we can rewrite the differential inequality (2.10),

$$h_0'' \ge -K|h_0'|, \tag{2.11}$$

which will hold whenever $h_0 \ge 0$. For later reference we also note that $h_0'(t) = D_\eta h(x_t)$ with vector η given by $\eta_j = E^{i,j}(q_i - q_{0i})$.

To prove Lemma 2.1, we take $y_0 = Tu(x_0)$, which implies that the function $G_0 = G(x, y_0, z_0)$ is a local support near x_0, If $h(x) < 0$, that is $u(x) < G(x, y_0, z_0)$, then we must have $h(x_{t_1}) > 0$ for some $t_1 \in (0,1)$, which implies that h_0 takes a positive maximum at some $t_2 \in (0,1)$. Clearly this contradicts the inequality (2.11). Therefore $h(x) \ge 0$ whence G_0 is a global support in Ω and Lemma 2.1 follows. □

Note that once we have $h_0 \ge 0$, we then have that (2.11) holds everywhere and this implies a gradient estimate,

$$0 \le (1-t)D_\eta h(x_t) \le Ch(x) \tag{2.12}$$

for some positive constant C, depending on G, Ω and $Tu(\Omega)$, which extends the fundamental lemma in [29].

To prove Lemma 2.2, we first note that a G-convex function u is semi-convex so that at any singular point x_0, its subgradient $\partial u(x_0)$ is a convex set whose extreme points are limits of points of differentiability. The result then follows by showing that the image $P_0 := G_x[(x_0, \cdot, H(x_0, \cdot, u_0)]Tu(x_0)$, where $u_0 = u(x_0)$, is convex in $\mathbb{R}^n$, (that is $Tu(x_0)$ is G^*-convex with respect to (x_0, u_0)). Accordingly we fix two points $y_1, y_2 \in Tu(x_0)$ and define corresponding G-affine functions, $u_i(x) = G(x, y_i, z_i)$ for $z_i = H(x_0, y_i, u_0), i = 1, 2$. Then for any p_0, lying in the interior of the straight line segment joining $Du_1(x_0)$ and $Du_2(x_0)$, and $y_0 = Y(x_0, u_0, p_0)$, either $u = u_1$ or $u = u_2$ satisfies the condition, $h_0'(0) = D_\eta h(x_0) > 0$, with respect to a fixed x in Ω or $D_\eta u_1(x_0) = D_\eta u_2(x_0)$. In the first cases, we obtain $h(x_{t_1}) > 0$ for some t_1 in $(0,1)$ close to 0 so that the proof of Lemma 2.1 is again applicable and we infer $G(x, y_0, z_0) \le \max\{u_1, u_2\}(x) \le u(x)$ whenever $\eta.(Du_1 - Du_2)(x_0) \ne 0$. By approximation we obtain then $G(x, y_0, z_0) \le u(x)$ for all $x \in \Omega$ and we conclude $y_0 \in Tu(x_0)$ as required. □

Lemma 2.3 follows immediately from the differential inequality (2.10), if $u \in C^2(\Omega)$. Otherwise, we may argue along the lines of [29, 31]. Assuming $S_\sigma \subset\subset \Omega$ and $\sigma > 0$, for any point $x_1 \in \partial S_\sigma, u_1 = u(x_1), y_1 \in Tu(x_1), z_1 = H(x_1, y_1, u_1)$, the inequality,

$$G(x, y_1, z_1) < G(x, y_0, z_0) + \sigma, \tag{2.13}$$

holds for all $x \in S_\sigma$. Making the transformation $x \to q = Q(x, y_0, z_0)$ as above, we need to show the transformed domain $Q_0 = Q(\cdot, y_0, z_0)(\Omega)$ is convex. If Q_0 is not convex, there must be a straight line segment ℓ joining two points in Q_0, containing a boundary point $q_1 = q(x_1)$. Now we choose

$$h(x) = G(x, y_1, z_1) - G(x, y_0, z_0) \tag{2.14}$$

and apply the differential inequality (2.10) along ℓ. As before we see that h cannot take a positive maximum on ℓ and hence $\ell \subset S_\sigma$. The case $\sigma = 0$ follows since $S_0 = \cap_{\sigma>0} S_\sigma$ and Lemma 2.3 is proved. □

When the function u is strictly G-convex, that is $Tu(x_0)$ is a single point for each $x \in \Omega$, then the above argument shows that S_σ for $\sigma > 0$ is strictly G-convex with respect to $y_0 = Tu(x_0)$, that is the set $Q(\cdot, y_0, z_0)(S_\sigma)$ is strictly convex.

In the optimal transportation case, $G(x, y, z) = c(x, y) - z$, the c-convexity of sections is proved and used in [5, 15].

To conclude this section we note that we have used condition G4w to ensure that the differential inequality (2.10) at least holds when $h \geq 0$. Otherwise it should be restricted to the set where $h = 0$. In this case we can still prove versions of our Lemmas by strengthening other hypotheses. We will take up these results in a sequel [27].

3. **Duality.** In this section we derive the dual Monge-Ampère equation for the G-transform v of an elliptic solution u of the prescribed Jacobian equation associated with a generating function G and prove the invariance of conditions G3, G3w under duality. Let $u \in C^2(\Omega)$ be elliptic for (1.4) (1.14) and suppose that the mapping Tu defined by (1.7) is a diffeomorphism from to Ω to the target domain Ω^*, where G satisfies G1,G2,G1* and $\Omega \times \Omega^* \subset \mathcal{U}$. Then we define the local G-transform of u by

$$\begin{aligned} v(y) = u^*_{G,loc}(y) &= H\big(T^{-1}y, y, u \circ T^{-1}(y)\big) \\ &= Z(\cdot, u, Du) \circ (T^{-1}(y) \end{aligned} \tag{3.1}$$

where $H = G^*$ is the dual generating function introduced in (1.17) and Z is defined by (1.13). When u is G-convex in Ω, then $u^*_{G,loc}$ agrees with the G-transform defined in (1.21). From (3.1) we have, for $y \in \Omega^*$,

$$Dv(y) = -\frac{G_y}{G_z}\big(T^{-1}y, y, v(y)\big) \tag{3.2}$$

and hence

$$T^{-1} = X(\cdot, v, Dv) \tag{3.3}$$

by G1*, where X is defined by (2.5). Consequently if u is an elliptic solution of the prescribed Jacobian equation (1.1), (1.19), then v is an elliptic solution of the dual prescribed Jacobian equation,

$$\det DX(\cdot, v, Dv) = \psi^*(\cdot, v, Dv), \quad |\psi^*| = g/f \circ X(\cdot, v, Dv), \tag{3.4}$$

that is

$$\det\big[D^2 v - A^*(\cdot, v, Dv)\big] = B^*(\cdot, v, Dv) \tag{3.5}$$

where

$$\begin{aligned} A^*(y,z,q) &= H_{yy}\left[X, y, u(X)\right] \\ &= -\left[\left(\frac{G_y}{G_z}\right)_y (X,y,z) + \left(\frac{G_y}{G_z}\right)_z (X,y,z) \otimes q\right], \end{aligned} \tag{3.6}$$

$$\begin{aligned} B^*(y,z,q) &= \big|\det H_{x,y}\big| \frac{g}{f \circ X} \\ &= \left(-\frac{1}{G_z}\right)^n \big|\det E\,\big| \frac{g}{f \circ X}, \end{aligned}$$

satisfying the dual second boundary value problem:

$$T^* v(\Omega^*) := X(\cdot, v, Dv)(\Omega^*) = \Omega. \tag{3.7}$$

We can then formulate the dual conditions,

G3* (G3*w) $\qquad D_{q_i q_j} A^*_{kl}\, \xi_i \xi_j\, \eta_k \eta_\ell \;>, (\geq)\; 0,$

for all $(X, y) \in \mathcal{U}, z \in I(X,y), \xi, \eta \in \mathbb{R}^n$, $\xi \cdot \eta = 0$.

Theorem 3.1. *Conditions $G3, (G3w)$ and $G3^*, (G3^*w)$ are equivalent.*

Proof. It will be convenient where possible to express our formulae in terms of the matrix function $[E_{i,j}]$ defined in Condition $G2$ and the vector function $[Q_{,i}]$ given by $Q_{,i} = q_i = -\frac{G_{y_i}}{G_z}$ in Condition $G1^*$. Partial derivatives of these quantities will then align with our notation in (2.2). As already indicated in Section 1, we have $Q_{i,j} = -E_{i,j}/G_z$. Accordingly we can express the last formula in (2.3) in the form

$$\begin{aligned} D_{p_k} A_{ij} &= E^{r,k}\big(E_{ij,r} - Q_{j,r} G_{i,z}\big) \\ &= E^{r,k}\big(E_{ij,r} + \frac{E_{j,r}}{G_z} G_{i,z}\big) \\ &= E^{r,k} E_{ij,r} + \frac{G_{j,z}}{G_z}\,\delta_{i,k}. \end{aligned} \tag{3.8}$$

Now differentiating, with respect to p_ℓ, we obtain

$$D^2_{p_k p_\ell} A_{ij} = \big(D_{p_\ell} E^{r,k}\big) E_{ij,r} + E^{r,k} D_{p_\ell}\big(E_{ij,r}\big) + D_{p_\ell}\big(\frac{G_{i,z}}{G_z}\big)\delta_{jk} \tag{3.9}$$

so that using (2.3) again we have

$$\begin{aligned} A^{k\ell}_{ij} = E^{r,k} E^{s,\ell}\big\{E_{ij,rs} + E_{ij,r,z} Q_{,s} - E^{r',s'} E_{ij,r'}\big(E_{s',rs} + E_{s',r,z} Q_{,s}\big)\big\} \\ + D_{p_\ell}\big(\frac{G_{i,z}}{G_z}\big)\delta_{jk} \end{aligned} \tag{3.10}$$

To get the corresponding formula for A^*, we first write,

$$Q_{r,k\ell} = -\frac{1}{G_z}\big(E_{r,k\ell} - \frac{E_{r,k}}{G_z} G_{,\ell,z}\big)$$

and

$$Q_{r,k,z} = -\frac{1}{G_z}\big(E_{r,k,z} - \frac{E_{r,k}}{G_z} G_{zz}\big).$$

Then we have

$$\begin{aligned} D_{q_i} A^*_{k\ell} = E^{i,r}\big(E_{r,k\ell} + E_{r,k,z} Q_{,\ell}\big) - \frac{1}{G_z}\big(G_{,\ell,z} + G_{zz} Q_{,\ell}\big)\delta_{ik} \\ + Q_{,k,z}\delta_{i\ell}. \end{aligned} \tag{3.11}$$

Consequently we obtain the dual formula,

$$\begin{aligned} D^2_{q_i q_j} A^*_{k\ell} = -G_z E^{i,r} E^{j,s}\Big\{E_{rs,k\ell} + E_{ij,k,z} Q_{,\ell} - E^{r',s'} E_{rs,r'}\big(E_{s',k\ell} + E_{s',k,z} Q_{,\ell}\big)\Big\} \\ + E^{i,r} E_{r,k,z}\delta_{j\ell} - D_{q_j}\Big[\frac{1}{G_z}\big(G_{,\ell,z} + G_{zz} Q_{,\ell}\big)\Big]\delta_{ik} + D_{q_j}\big(Q_{,k,z}\big)\delta_{i\ell}. \end{aligned} \tag{3.12}$$

Formulae (3.10) and (3.12) will hold for all $(x, y, z) \in \Gamma$ and imply the equivalence of conditions G3(G3w) and G3*(G3*w), using Condition $G2$, (that is $\det E \neq 0$), and the orthogonality of ξ and η. □

4. **Existence and regularity.** We begin with the definition of generalized solution. Let Ω and Ω^* be bounded domains in $\mathbb{R}^n$, with $\overline{\Omega} \times \overline{\Omega}^* \subset \mathcal{U}$, and let $u \in C^0(\overline{\Omega})$ be G-convex in Ω, with conditions G1,G2, G1* satisfied. Following [21], we introduce a measure $\mu = \mu_g[u]$ on Ω, for $g \geq 0 \in L^1(\Omega)$, such that for any Borel set $E \subset \Omega$,

$$\mu(E) = \int_{Tu(E)} g \tag{4.1}$$

To prove that μ is a Radon measure, we can extend the argument in [21], using the G-transform v, defined in (1.21), namely

$$v(y) = u^*_G(y) = \sup_\Omega H(\cdot, y, u).$$

Note that v will be G^*- convex in Ω^*. Defining the dual G^*-transform on Ω^*, v^*, by

$$v^*(x) = \sup_{y\in\Omega^*} G(x, y, v(y)), \tag{4.2}$$

we obtain $v^* = u$ in Ω and furthermore $y \in Tu(x)$ if and only if $x \in T^*v(y)$, where T^* denotes the G^*-normal mapping on Ω^*. Since the functions u and v are semi-convex and hence twice differentiable almost everywhere in Ω and Ω^*, respectively, we then infer that the subset of points $y \in \Omega^*$, such that $y \in Tu(x_1) \cap Tu(x_2)$ for some $x_1 \neq x_2, \in \Omega$, has measure zero and from this it follows that μ is countably additive. Also extending the argument in [21], we have that μ is weakly continuous with respect to local uniform convergence, namely if $\{u_m\}$ is a sequence of G-convex functions in Ω, converging to u, then the sequence of measures $\{\mu_g[u_m]\}$ converges to $\mu_g[u]$ weakly. These arguments also parallel the special case of the Monge-Ampère measure of Aleksandrov, as presented for example in the book [8].

A G-convex function u on Ω is now defined to be a generalized solution of the second boundary value problem (1.7) for equation (1.1), (1.8), under the mass balance condition (1.9), if

$$\mu_g[u] = \nu_f \tag{4.3}$$

where $\nu_f = f\mathrm{d}x$ and g is extended to vanish outside Ω^*. More generally we can replace ν_f by any finite Borel measure ν on Ω. We remark that this notion corresponds to that of generalized solution of Type A in [12], where a generalized solution of Type B corresponds to the dual notion,

$$\mu^*[u](E^*) := \int_{T^{-1}(E^*)} f = \nu^*(E) \tag{4.4}$$

for any Borel set $E^* \subset \Omega^*$, that is $\mu_f[v] = \nu^*$, where $\nu^* = g\,\mathrm{d}y$. As in the optimal transportation case in [21], the two notions are equivalent provided the measures ν and ν^* have densities $f \in L^1(\Omega)$, $g \in L^1(\Omega^*)$ respectively. Therefore we have:

Lemma 4.1. *Let u be a generalized solution of the second boundary value problem (1.7) for equation (1.1), (1.8), under the mass balance condition (1.9), and let v be the G-transform of u. Then v is a generalized solution of the second boundary value problem (3.4), (3.7).*

Next we formulate an existence theorem under condition G5 which extends the optimal transportation case.

Theorem 4.2. *Let Ω and Ω^* be bounded domains in $\mathbb{R}^n$, with $\overline{\Omega} \times \overline{\Omega}^* \subset \mathcal{U}$ and let G be a generating function satisfying G1,G2,G1* and G5. Suppose that f and g are positive densities in $L^1(\Omega)$ and $L^1(\Omega^*)$ satisfying the mass balance condition (1.9).Then for any $x_0 \in \Omega$ and $u_0 > m_0 + K_1$, where $K_1 = K_0\mathrm{dist}(x_0, \partial\Omega)$, there exists a generalized solution of (1.4),(1.7) satisfying $u(x_0) = u_0$.*

The proof of Theorem 4.2 follows the approach in [3, 11] using approximations by piecewise G-affine functions to solve the dual problem (4.4), where ν^* is approximated by a linear combination of Dirac delta measures. Condition G5 implies that a generalized solution u whose graph passes through (x_0, u_0) satisfies $u \geq m_1$ for some $m_1 > m_0$, together with an a priori bound:

$$|u|_{0,1} \leq K, \tag{4.5}$$

with constant K depending on K_0, u_0 and diam Ω. The solvability of the dual problem, with an arbitrary finite measure ν^* then follows from the weak continuity of μ^*. Theorem 4.2 then follows from Lemma 4.1. □

Taking account of our remark following the formulation of G5 in Section 1 and choosing $\mu(G) = \log(G)$ with $m_0 = 0$, we see that if we replace the gradient bound in G5 by

$$|G_x(x, y, z)| \leq K_0 G(x, y, z), \tag{4.6}$$

we conclude that there exists a generalized solution u such that $u(x_0) = u_0$ for any given $u_0 > 0$. We remark also that instead of prescribing $u(x_0)$ in Theorem 4.3, we may fix instead $u_{min} = m_1 > m_0$.

The theory developed in the preceding sections can now be applied to prove interior regularity results for generalized solutions. A crucial lemma for this purpose is Lemma 2.2 on the characterization of the mapping Tu by the subdifferential ∂u, which shows that the concept of generalized solution is local, that is it is the same for subdomains. This property was originally overlooked in [21] but rectified subsequently in [31], with a more direct approach found in [10]. In this paper we will just prove interior smoothness under condition G3 following the approach in [21]. Further extensions of optimal transportation regularity depend also on Lemma 2.3 on the G-convexity of sections but we will not pursue these here. First we state some more lemmas which extend the corresponding results in [21].

Lemma 4.3. *Let u be a generalized solution of (1.4), (1.14),(1.7). Suppose $f > 0$ in Ω and Ω^* is G-convex with respect to u. Then $Tu(\Omega) \subset \overline{\Omega}^*$.*

Lemma 4.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $u, v \in C^0(\overline{\Omega})$, G-convex functions satisfying $u \geq v$ in Ω and $u = v$ on $\partial\Omega$. Then $Tu(\Omega) \subset Tv(\Omega)$.*

The proofs of Lemmas 4.3 and 4.4 are essentially identical with the optimal transportation case in [21]. Lemma 4.3 makes critical use of the subdifferential being convex in general. To prove Lemma 4.4 we fix some $x_0 \in \Omega$, $y_0 \in Tu(x_0)$ and increase $z_0 = H(x_0, y_0, u(x_0))$ in I until the function $G(\cdot, y_0, z_1)$, for $z_1 > z_0$, becomes a G-support for v at some point $x_1 \in \Omega$. But then, using Lemma 2.2 if v is not differentiable at x_1, we must have $y_0 \in Tu(x_1)$ since v is G-convex. □

Now we formulate the interior regularity result, which extends the main result in [21].

Theorem 4.5. *Let u be a generalized solution of (1.7) with positive densities $f \in C^{1,1}(\Omega), g \in C^{1,1}(\Omega^*)$ with $f, 1/f \in L^\infty(\Omega), g, 1/g \in L^\infty(\Omega^*)$ and with generating function G satisfying conditions, G1,G2,G1*, G3 and G4w. Suppose that Ω^* is G^*-convex with respect to u. Then $u \in C^3(\Omega)$ and is an elliptic solution of (1.4), (1.14). Furthermore if Ω is G-convex with respect to $v = u^*_G$, then Tu is also a diffeomorphism from Ω to Ω^*, with v an elliptic solution of the dual equation (3.5).*

To prove Theorem 4.5 by the method in [21], we need to solve approximating Dirichlet problems,

$$\begin{aligned} \det[D^2 w - A(\cdot, w, Dw)] &= B(\cdot, w, Dw) \quad \text{in } B_r, \\ w &= u_m \qquad \text{on } \partial B_r, \end{aligned}$$

where B_r is a small ball in a fixed subdomain $\Omega' \subset\subset \Omega$ and $\{u_m\}$ is a sequence of smooth functions converging uniformly in Ω' to u. Specifically, we set

$$u_m = \tilde{u}_{h_m} - c_0|x|^2,$$

where $\tilde{u} = u + c_0|x|^2$ is convex in Ω' and $\tilde{u}_{h_m}$ denotes the mollification of $\tilde{u}$ with $h_m \to 0$. It follows also from the local Lipschitz continuity and semi-convexity of the G-convex function u, then that for m sufficiently large, the constant c_o can be chosen so that

$$(|u_m| + |Du_m|) \leq c_o, \quad D^2 u_m \geq -c_0 I$$

in Ω' and moreover u_m is admissible in the sense that

$$(x, u_m(x), Du_m(x)) \in \Gamma'(\Omega) \tag{4.7}$$

for all $x \in \Omega'$, where in accordance with our notation in Section 1,

$$\Gamma'(\Omega) = \big\{(x,u,p) \mid x \in \Omega, u = G(x,y,z), p = G_x(x,y,z),$$
$$\text{for some } y \in \mathcal{U}^*_x, z \in I(x,y)\big\}.$$

The essential difference here with the corresponding equations in [21] is the dependence of the matrix function A and scalar function B on u. The necessary existence and uniqueness result is expressed in the following lemma.

Lemma 4.6. *Let G satisfy conditions G1, G2, and G3w in $\mathcal{U}$, $\Omega \subset \mathcal{U}^{(1)}$ and suppose $B > 0, \in C^{1,1}(\Gamma')$. Let B_r be a small ball in $\Omega' \subset\subset \Omega$ and suppose $u_m \in C^4(\Omega')$ satisfies the constraint (4.7). Then for sufficiently small r, depending on G, B and c_o, there exists a unique elliptic solution $w \in C^3(\bar{B}_r)$ of the Dirichlet problem (4.7), also satisfying (4.7) in B_r together with an a priori C^1 bound,*

$$|w| + |Dw| \leq C, \tag{4.8}$$

with constant C also depending only on G, B and c_0.

To prove Lemma 4.6 by the method of continuity, (as in [6]), we need suitable a priori estimates for solutions and their derivatives up to second order. For simplicity we can assume $B_r = B_r(0)$ is centred at the origin. First we note that, even with the dependence of A and B on the solution w, the functions

$$v = v_m = u_m + k(|x|^2 - r^2)$$

will still be strict elliptic upper barriers for (4.7) for sufficiently large k and small r, depending on A, B and c_0, as well as satisfy (4.7) in B_r. Moreover for any constant c_1, we can obtain by such choice the strong differential inequality,

$$\det[D^2 v - A(\cdot, v, Dv)] > B(\cdot, v, Dv) + c_1 \quad \text{in } B_r.$$

Now let w be an elliptic solution of (4.7) satisfying (4.7) in B_r and fix $k = k_0$ and $r = r_0$ as above. We claim then that for $k = k_0$ and sufficiently smaller $r < r_0$, we have $w \geq v$ in B_r. To show this we suppose $M = \max(v - w) > 0$ is taken on at $x_0 \in \Omega$. Since $Dv(x_0) = Dw(x_0)$ and $D^2 v(x_0) \leq D^2 w(x_0)$ we obtain, by increasing k appropriately,

$$w - v \leq CMr^2$$

for further constant C depending on A, B, k_0, r_0 and c_0. Choosing r sufficiently small, we infer $w \geq v$, as claimed. We obtain thus an estimate for w from below. Furthermore we can also obtain, since v is convex for $k \geq c_o$,

$$w > v, \quad Dw(\partial B_r) \subset Dv(\bar{B}_r) \tag{4.9}$$

which provides an estimate for Dw on ∂B_r. Next since the Jacobian determinant, $\det DTw \neq 0$ in B_r, we see that $|Tw|^2$ takes its maximum on ∂B_r so that we obtain an estimate in B_r for Tw. To complete the gradient bound (4.8), we need an estimate for w from above. Here we use the ellipticity to obtain

$$\triangle w \geq \text{traceA}(\cdot.w, Dw)$$

so that with a similar argument we can show

$$w \leq \underline{v} = \text{sup}u_m - k(|x|^2 - r^2)$$

in B_r for sufficiently large k and small r. From the estimates for w and Tw, we then have an estimate for Dw in B_r and so (4.8) is proved.

We remark that if G1* and G4 are also satisfied then both w and v will be G-convex in B_r by Lemma 2.1 and we can use Lemma 4.4 to obtain $Tw(B_r) \subset Tv(B_r)$ and hence alternatively estimate Tw in B_r.

Global second derivative estimates for solutions of (4.7) follow as in the case where there is no dependence on u in A and B, since functions φ given by

$$\varphi(x) = k(|x|^2 - r^2) \tag{4.10}$$

provide barriers for the linearized operators in B_r for sufficiently large k and small r, that is

$$[D_{ij}\varphi - D_{p_k}A_{ij}(\cdot, w, Dw)D_k\varphi]\xi_i\xi_j \;\geq\; |\xi|^2 \tag{4.11}$$

in B_r, for all $\xi \in \mathbb{R}^n$. The reader is referred to the papers [9, 23, 30] for more details.

The uniqueness of solutions of (4.7) can be shown by the same comparison argument as above or by using the linearized equation and (4.11). Once second derivative bounds and uniqueness are established, the standard method of continuity in [6] is applicable with the smallness of the radius r also used to imply the invertibility of the linearized operators in the associated deformation. □

In order to proceed from Lemma 4.6 we send m to ∞, that is for u_m to approach u. At this stage we need to replace G3w by the strict regularity condition G3 to obtain an interior second derivative estimate for solutions, namely for any $r' < 1$

$$|D^2 w| \;\leq\; C \tag{4.12}$$

in $B_{r'}$ where C depends on $n, A, B, \sup_{B_r}(|w| + |Dw|)$ and $r - r'$. In the optimal transportation case this was the key estimate in [21] and as remarked there it also embraces general equations of the form (1.4); (see [29] for a direct proof).

Using also the gradient estimate (4.8) we then conclude that the solutions w_m of (4.7) converge to a unique solution $w \in C^3(B_r) \cap C^{0,1}(\bar{B}_r)$ of the limiting problem $w = u$ on ∂B_r. To complete the proof of Theorem 4.5, we need to show that $w = u$. This is more delicate than in the optimal transportation case as we cannot localise around maximum or minimum points but we overcome this obstacle by using both Lemmas 2.1 and 2.2. Let us first suppose that $w - u$ takes a positive maximum M at some point x_0 in B_r. Since u and w lie in $C^{0,1}(\bar{B}_r)$ there exists a radius $r' < r$, depending on M and the Lipschitz constants of u and w, such that $w - u < M/2$ on $\partial B_{r'}$. Since $w \in C^2(\bar{B}_{r'})$, for sufficiently small $\epsilon > 0$, we can perturb w to get a function $w_\epsilon \in C^2(\bar{B}_{r'})$, satisfying $|w - w_\epsilon| < \epsilon$, which is a strict upper barrier for (4.7), that is

$$\det[D^2 w_\epsilon - A(\cdot, w_\epsilon, Dw_\epsilon)] > B(\cdot, w_\epsilon, Dw_\epsilon) \tag{4.13}$$

in $B_{r'}$. Furthermore w_ϵ is elliptic for (4.7) and satisfies (4.7) in $B_{r'}$. Hence for small enough $\epsilon < M/4$ the subdomain $\Omega_\epsilon = \{w_\epsilon > u\} \subset B_{r'}$ and moreover by Lemma 2.1, w_ϵ is G-convex in Ω_ϵ so that from (4.13),

$$\begin{aligned}\mu_g[w_\epsilon](\Omega_\epsilon) &> \int_{\Omega_\epsilon} f \\ &= \mu_g[u](\Omega_\epsilon)\end{aligned} \tag{4.14}$$

by Lemma 2.2. Now using the monotonicity, Lemma 4.3, we reach a contradiction and hence $w \le u$ in B_r. Similarly we show $w \ge u$ in B_r, whence $w = u$ in B_r and consequently $u \in C^3(\Omega)$ as asserted. □

We point out however that we have implicitly used a stronger condition on the target density g in the proof of Theorem 4.5, namely $g \in C^{1,1}(\overline{\Omega}^*)$. To use only the local smoothness of g we need to prove in advance that $u \in C^1(\Omega)$, (to keep $T(B_r)$ away from $\partial\Omega^*$), or equivalently by duality that u is strictly G-convex.This may be accomplished under the hypotheses $f, 1/f \in L^\infty(\Omega), g, 1/g \in L^\infty(\Omega^*)$ as in [30] or [14] and will be treated in [27] in conjunction with the relaxation of condition G4w. Moreover, as in [14], we also infer $u \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$.

Under our smoothness conditions on f and g in Theorem 4.5, we actually have the solution $u \in C^{3,\alpha}$ for all $\alpha < 1$, by the Schauder theory [6]. We also obtain that if $f \in C^\infty(\Omega), g \in C^\infty(\Omega^*)$, then $u \in C^\infty(\Omega)$. As in the case with optimal transportation, regularity results may be refined when perturbation arguments using Lemma 2.3 are employed instead of Lemma 2.2. In particular, when we replace $C^{1,1}$ by $C^{0,\alpha}$ in Theorem 4.5, we obtain from [20] that $u \in C^2(\Omega)$. We will not pursue this approach in this paper. It would be interesting though to have extensions to G3w along the lines of [5, 33], that is to show solutions are strictly G-convex and continuously differentiable under G3w, together with appropriate domain convexity conditions. We remark also that using the interior second derivative estimates in [18], it follows from our proof of Theorem 4.5 that condition G3 in the hypothesis can be relaxed to G3w if the solution u is assumed strictly G-convex in Ω.

We also remark here that if condition G3w is violated then Lemma 2.2 is no longer true in general and that generalized solutions are not necessarily continuously differentiable. This follows in the same way as the optimal transportation case in [14, 21]. Similarly from [21], the G^*-convexity of the target Ω^* is also necessary for regularity. An interesting consequence is that if the generating function satisfies G3, then generalized solutions of the complementary problem associated with condition (1.16) cannot be continuously differentiable in general.

Combining Theorems 4.2 and 4.5 we get an existence result for locally smooth solutions.

Corollary 4.7. *Let Ω and Ω^* be bounded domains in $\mathbb{R}^n$, with $\overline{\Omega} \times \overline{\Omega}^* \subset \mathcal{U}$ and let G be a generating function satisfying G1,G2,G1*,G3,G4w and G5. Suppose that $f \in C^{1,1}(\Omega), g \in C^{1,1}(\Omega^*)$ with $f, 1/f \in L^\infty(\Omega), g, 1/g \in L^\infty(\Omega^*)$ and that f and g satisfy the mass balance condition (1.9).Then for any $x_0 \in \Omega$ and $u_0 > m_0 + K_1$, there exists a G-convex, elliptic solution $u \in C^3(\Omega)$of (1.4),(1.7) satisfying $u(x_0) = u_0$ and $Tu(\Omega) = \Omega^*$ a.e., provided Ω^* is G^*-convex with respect to $\Omega \times (m_0, \infty)$. If also Ω is G-convex with respect to $\Omega^* \times I$, then Tu is a diffeomorphism from Ω onto Ω^*.*

We remark that the interval $(m_0.\infty)$ in the G^*-convexity hypothesis may be replaced by the subinterval, $(u_0 - K_1, u_0 + K_1)$, while to ensure Tu is onto Ω^* we need only assume that Ω is G-convex with respect to all $y \in \Omega^*, z \in I$ satisfying

$$|G(\cdot, y, z)(\Omega) - u_0| < K_1.$$

Furthermore if (4.6) holds then we can take any $u_0 > m_0 = 0$ in Corollary 4.7. In [27], we show that condition G4w is not needed for Corollary 4.7 as the other hypotheses there will still enable the applications of Lemmas 2.1 and 2.2.

Theorems 4.2,4.5 and Corollary 4.7 clearly apply to the parallel beam example (1.19) and under the hypotheses of Corollary 4.7 we would obtain the existence of a positive elliptic solution u of equation (1.20) with $|Du| < 1$.

We conclude this section with some remarks about the point source reflection problem treated in [11, 12, 19]. Invoking the ellipsoid of revolution in its polar form [11, 12] and projecting onto the unit ball in $\mathbb{R}^n$, we may take $\mathcal{U} = B_1(0) \times \mathbb{R}^n, I = (0, \infty)$. In particular, if the target hypersurface lies in a graph $y_{n+1} = T(y)$ in $\mathbb{R}^{n+1}$, we may model the reflection process by a generating function:

$$G(x, y, z) = \frac{1}{z}\big\{\sqrt{z + |y|^2 + T^2} - x.y - \sqrt{1 - |x|^2}\ T\big\}. \tag{4.15}$$

In (4.15), the parameter $z = (2b)^2$, where b is the minor axis of the supporting ellipsoid of revolution $E = E(y, y_{n+1}, z)$, with foci at 0 and (y, y_{n+1}), and E is the graph of $\rho = 1/2G$ over the upper hemisphere.

Differentiating (4.15) we obtain,

$$\begin{aligned} G_x &= \frac{1}{z}(-y + \frac{x}{\sqrt{1 - |x|^2}}\ T), \\ G_{xx} &= \frac{1}{z(1 - |x|^2)^{3/2}}(I + x \otimes x)T. \end{aligned} \tag{4.16}$$

Let us restrict here to a special case where the target lies in a hyperplane not above the source at the origin, namely $y_{n=1} = \tau \le 0$. Writing $\bar{u} = u - p.x, p = G_x$, it follows that $\bar{u} > |p|$ and moreover we have the formulae:

$$\begin{aligned} Z &= (1 - \frac{2\tau u}{\sqrt{1 - |x|^2}})/(\bar{u}^2 - |p|^2), \\ Y &= -pZ + \frac{\tau x}{\sqrt{1 - |x|^2}}, \\ A &= \frac{\tau(\bar{u}^2 - |p|^2)}{(1 - |x|^2)(\sqrt{1 - |x|^2} - 2\tau u)}(I + x \otimes x). \end{aligned} \tag{4.17}$$

It is now easy to check that G satisfies G1,G2,G1*,G2,G3,G4 with $I(x, y) = J(x, y) = (0, \infty)$ if $\tau < 0$, and G1,G2,G1*,G3w,G4w for $\tau = 0$, in which case $A = 0$. Furthermore if $\bar{\Omega} \subset B_1(0)$, then G also satisfies (4.6). The resultant existence and regularity results correspond to special cases of [12] except we are also able to solve the boundary value problem (1.7) in the classical sense when Ω is also G-convex. However the local regularity results in [12] are more general in that condition G3 need not be satisfied everywhere for more general targets and in this case a stronger version of Lemma 2.2 enables local regularity to be proved in the sub-domains where G3 is satisfied. When $\tau = 0$, G-convexity coincides with convexity and local regularity under G^*-convexity of the target domain Ω^* follows from [1], Lemma 3.

Finally we note that the existence of globally smooth solutions of the point source and parallel beam near field reflection problems is established in [19] and that solutions of these and related problems can also be represented as potentials of a nonlinear Kantorovich problem, [7, 16, 17].

REFERENCES

[1] L. Caffarelli, *The regularity of mappings with a convex potential*, J. Amer. Math. Soc., **5** (1992), 99–104.

[2] L. Caffarelli, *Boundary regularity of maps with convex potentials II*, Problems in mathematical analysis. No. 37. Ann. of Math., **144** (1996), 453–496.

[3] L. Caffarelli and V. I. Oliker, *Weak solutions of one inverse problem in geometric optics*. J. Math. Sci. (N. Y.), **154** (2008), 39–49.

[4] Ph. Delanoë, *Classical solvability in dimension two of the second boundary value problem associated with the Monge-Ampère operator*, Ann. Inst. Henri Poincaré, Analyse Non Linéaire, **8** (1991), 443–457.

[5] A. Figalli, Y.-H Kim and R. J. McCann, *Hölder continuity and injectivity of optimal maps*, Arch. Ration. Mech. Anal., **209** (2013), 747–795.

[6] D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224. Springer-Verlag, Berlin, 1983.

[7] T. Graf and V. Oliker, *An optimal mass transport approach to the near-field reflector problem in optimal design*, Inverse Problems, **28** (2012), 1–15.

[8] C. E. Gutierrez, "The Monge -Ampère Equation," Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston, Inc., Boston, MA, 2001.

[9] F. Jiang, N. S. Trudinger and X.-P. Yang, *On the Dirichlet Problem for Monge-Ampère type equations*, Calc. Var. Partial Differ. Equ. , to appear, online DOI 10.1007/s00526-013-0619-3.

[10] Y.-H. Kim and R. J. McCann, *Continuity, curvature and the general covariance of optimal transportation*, J. Eur. Math. Soc., **12** (2010), 1009–1040.

[11] S. A. Kochengin and V. I. Oliker, *Determination of reflector surfaces from near-field scattering data*, Inverse Problems, **13** (1997), 363–373.

[12] A. Karakhanyan and X.-J. Wang, *On the reflector shape design*, J. Diff. Geom., **84** (2010), 561–610.

[13] P.-L. Lions, N. S. Trudinger and J. Urbas, *Neumann problem for equations of Monge-Ampère type*, Comm. Pure Appl. Math., **39** (1986), 539–563.

[14] G. Loeper, *On the regularity of solutions of optimal transportation problems*, Acta Math., **202** (2009), 241–283.

[15] J.-K. Liu, *Hölder regularity of optimal mappings in optimal transportation*, Calc. Var. Partial Differ. Equ., **34** (2009), 435–451.

[16] J.-K. Liu, *Light reflection is nonlinear optimization*, Calc. Var. Partial Differ. Equ., **46** (2013), 861–878.

[17] J.-K. Liu, *A class of nonlinear optimization problems with potentials*, preprint (2012).

[18] J.-K. Liu and N. S. Trudinger, *On Pogorelov estimates for Monge-Ampère type equations*, Discrete Contin. Dyn. Syst., **28**, (2010), 1121–1135.

[19] J.-K. Liu and N. S. Trudinger, *On classical solutions of near field reflection problems*, preprint (2013).

[20] J.-K. Liu, N. S. Trudinger and X. J. Wang, *Interior $C^{2,\alpha}$ regularity for potential functions in optimal transportation*, Comm. Partial Differential Equations, **35** (2010), 165–184.

[21] X.-N. Ma, N. S. Trudinger and X.-J.Wang, *Regularity of potential functions of the optimal transportation problem*, Arch. Rat. Mech. Anal., **177** (2005), 151–183.

[22] S. T. Rachev and L. Ruschendorff, "Mass Transportation Problems," Springer, 1998.

[23] N. S. Trudinger, *Recent developments in elliptic partial differential equations of Monge-Ampère type*, International Congress of Mathematicians. Eur. Math. Soc., **III** (2006), 291–301.

[24] N. S. Trudinger, *On the prescribed Jacobian equation*, Gakuto Intl. Series, Math. Sci. Appl. **20** (2008), Proc. Intl. Conf. for the 25th Anniversary of Viscosity Solutions, 243–255.

[25] N. S. Trudinger, *On the local theory of prescribed Jacobian equations*, Conference in honor of 60th birthday of C.-S. Lin, Taipei, 2011, (`http://maths.anu.edu.au/~neilt/RecentPapers.html`).

[26] N. S. Trudinger, *On generated prescribed Jacobian equations*, Oberwolfach Reports, **38** (2011), 32–36.

[27] N. S. Trudinger, *The local theory of prescribed Jacobian equations revisited*, (in preparation).

[28] N. S. Trudinger and X.-J. Wang, *The Monge-Ampère equation and its geometric applications*, in "Handbook of Geometric Analysis," International Press (2008), 467–524.

[29] N. S. Trudinger and X.-J. Wang, *On convexity notions in optimal transportation*, preprint (2008).

[30] N. S. Trudinger and X.-J. Wang , *On the second boundary value problem for Monge-Ampère type equations and optimal transportation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. ,Series (5), **8** (2009), 143–174.

[31] N. S. Trudinger and X.-J.Wang, *On strict convexity and continuous differentiability of potential functions in optimal transportation*, Arch. Rat. Mech.Ȧnal. , **192** (2009), 403–418.

[32] J. Urbas, *On the second boundary value problem for equations of Monge-Ampère type*, J. Reine Angew. Math., **487** (1997),115–124.

[33] J. Vetois, *Continuity and injectivity of optimal maps*, preprint (2011).

[34] C. Villani, "Topics in Optimal Transportation," Graduate Studies in Mathematics, 58. American Mathematical Society, Providence, RI, 2003. xvi+370 pp.

[35] C. Villani, "Optimal Transportation, Old and New," Springer, 2008.

Received November 2012; revised May 2013.

E-mail address: neil.trudinger@anu.edu.au